A CLASS OF EXACT SOLUTIONS OF THE EQUATIONS OF STATICS FOR A GRANULAR MEDIUM

A. I. Dimov and A. F. Revuzhenko

UDC 539.374

A granular medium differs from a perfectly plastic medium by the presence of internal friction. In the limiting state, the maximum tangential stress τ is related to the normal stress by Coulomb's condition

$$\tau = -\frac{\sigma_1+\sigma_2}{2}\sin\rho + k\cos\rho, \tag{1}$$

where ρ is the angle of internal friction; k is the cohesiveness; σ_1, σ_2 are the principal stresses. Condition (1) closes the equations of equilibrium. This permits stating and solving statically determinable problems. A perfectly plastic medium is obtained as a particular case for $\rho \to 0$. The system of equations of equilibrium with (1) is a hyperbolic system [1]. As in any classical model, there is great interest in seeking exact solutions, which can be represented in closed form and investigated completely. One of the fundamental exact solutions is a solution in the form of a centered wave [1, 2], obtained as the limiting case of solutions in which one of the families of slip lines consists of straight lines. We shall look at this solution from a somewhat different point of view: we shall view the centered bundle of straight characteristics as radii of a polar system of coordinates. Then the solution is completely characterized by the single and very simple condition $\delta \equiv \pi/2 - \rho = \text{const}$, where δ is the angle between the slip lines and the polar radius. This interpretation leads immediately to the idea of finding new solutions by generalizing the condition $\delta \equiv \pi/2 - \rho = \text{const}$ [3]. It is first necessary to derive an equation in which only the angle δ enters. (At least, it is known that $\delta \equiv \pi/2 - \rho$ will be the exact solution of this equation.)

The first question that arises in this case is: which two variables must be taken as the independent variables? Since the angle δ relates the position of the slip lines and the polar radius, it is natural to choose the polar radii r and the angle θ as the independent variables. In this case, the well known exact solutions are obtained [4]. Another method involves transforming to characteristic coordinates λ_1, λ_2 and deriving an equation for δ in these coordinates. The closed system in the variables λ_1, λ_2 has the form

$$\begin{gathered}\cos\rho\frac{\partial\sigma}{\partial\lambda_1} + 2\sigma\sin\rho\frac{\partial\varphi}{\partial\lambda_1} = 0,\ \cos\rho\frac{\partial\sigma}{\partial\lambda_2} - 2\sigma\sin\rho\frac{\partial\varphi}{\partial\lambda_2} = 0,\\ \frac{\partial x_2}{\partial\lambda_1} = \operatorname{tg}\left(\varphi - \frac{\pi}{4} - \frac{\rho}{2}\right)\frac{\partial x_1}{\partial\lambda_1}, \quad \frac{\partial x_2}{\partial\lambda_2} = \operatorname{tg}\left(\varphi + \frac{\pi}{4} + \frac{\rho}{2}\right)\frac{\partial x_1}{\partial\lambda_2},\end{gathered} \tag{2}$$

where φ is the angle of inclination of the largest principle stress to the axis Ox_1; $\sigma = (\sigma_1 + \sigma_2)/2 - k \cot \rho$ and x_1 and x_2 are Cartesian coordinates.

Integrating the first two equations, we obtain

$$\varphi = \Phi_1(\lambda_1) + \Phi_2(\lambda_2),\ \ln|\sigma| = 2\operatorname{tg}\rho\,[\Phi_2(\lambda_2) - \Phi_1(\lambda_1)], \tag{3}$$

where Φ_1, Φ_2 are arbitrary functions. It follows from the definition of the angle δ that

$$\frac{\partial\ln r}{\partial\lambda_1} = -\operatorname{tg}(\delta-\rho)\frac{\partial\theta}{\partial\lambda_1},\ \frac{\partial\ln r}{\partial\lambda_2} = \operatorname{ctg}\delta\frac{\partial\theta}{\partial\lambda_2}. \tag{4}$$

If the arbitrary function $\delta(\lambda_1, \lambda_2)$ or $\delta(r, \theta)$ is fixed, then Eq. (4) will give some grid of lines λ_1, λ_2 with constant angle of intersection $\pi/2 - \rho$. Let us calculate at each point

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 164-167, May-June, 1983. Original article submitted May 18, 1982.

 0021-8944/83/2403-0442$07.50

the angle of inclination of the bisector between the lines λ_1, λ_2 relative to the Ox_1 axis. In order that this grid correspond to some stress distribution, it is necessary and sufficient that the angle of inclination of the bisector as a function of the coordinates λ_1, λ_2 have the form of a sum of some functions $\Phi_1(\lambda_1) + \Phi_2(\lambda_2)$. It is easy to make use of the last condition due to the simple relation between the angles:

$$\theta = \varphi - \delta + \pi/4 + \rho/2. \tag{5}$$

Then, substituting expression (5) for the angle θ into Eq. (4), eliminating r, and using the representation (3), we obtain finally

$$-\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}\ln\left|\frac{\cos(\rho-\delta)}{\sin\delta}\right| - \Phi_2'\frac{\partial \operatorname{ctg}\delta}{\partial\lambda_1} + \Phi_1'\frac{\partial \operatorname{tg}(\rho-\delta)}{\partial\lambda_2} = 0. \tag{6}$$

This second-order nonlinear equation is equivalent to the starting system. Then, the problem reduces to searching for exact solutions of Eq. (6). Its structure is such that it admits a solution for which the leading term vanishes:

$$\cos(\rho-\delta)/\sin\delta = -\xi_1(\lambda_1)/\xi_2(\lambda_2),$$

where ξ_1, ξ_2 are arbitrary functions.

From here we immediately find the angle δ as a function of λ_1, λ_2:

$$\delta = -\operatorname{arctg}\left(\frac{1}{\cos\rho}\frac{\xi_2}{\xi_1} + \operatorname{tg}\rho\right) + \rho,$$

and from Eq. (6)

$$\Phi_1 = c_1\xi_1^2 + c_2,\ \Phi_2 = -c_1\xi_2^2 + c_3.$$

It is now possible to obtain from Eq. (3)-(5) a solution of the entire system in parametric form in terms of the characteristic variables:

$$\begin{gathered}\varphi = c_1(\xi_1^2 - \xi_2^2) + c_2 + c_3,\ \sigma = -c_4\exp\left[-2c_1\operatorname{tg}\rho\,(\xi_1^2 + \xi_2^2)\right],\\ \theta = c_1(\xi_1^2-\xi_2^2) + \operatorname{arctg}\left(\frac{1}{\cos\rho}\frac{\xi_2}{\xi_1} + \operatorname{tg}\rho\right) - \frac{\rho}{2} + \frac{\pi}{4} + c_2 + c_3,\\ r = \frac{1}{\xi_2}\sqrt{1 + \left(\frac{1}{\cos\rho}\frac{\xi_2}{\xi_1} + \operatorname{tg}\rho\right)^2}\exp\left[\frac{c_1}{\cos\rho}\left(2\xi_1\xi_2 + \sin\rho\,(\xi_1^2+\xi_2^2)\right) + c_5\right],\end{gathered} \tag{7}$$

where $c_1 - c_5$ are constants of integration.

We have thus obtained a class of exact solutions of the starting system in terms of elementary functions. It is evident that the characteristic variables enter here only via ξ_1, ξ_2, so that the apparent enormous arbitrariness in the two functions in the representation (7) merely represents the arbitrariness in the parametrization of the characteristics. Therefore, the variables ξ_1, ξ_2 can be chosen as the parameters. Without loss of generality we can set $c_2 + c_3 = 0$, where these constants express the arbitrariness in choosing the origin for measuring the angles; $c_5 = 0$ and $|c_1| = 1$, since the variation of these coordinates corresponds to a stretching transformation of the spatial variables and parameters ξ_1, ξ_2. We finally obtain two classes of solutions, corresponding to the value $c_1 = +1$ or $c_1 = -1$. We shall examine further only the last case, since the solution with $c_1 = +1$ is analogous.

In order that the mapping $(\xi_1, \xi_2) \to (x_1, x_2)$ be one-to-one, it is necessary that $\Delta = \partial(r, \theta)/\partial(\xi_1, \xi_2) \neq 0$ everywhere in the region of variables (ξ_1, ξ_2) mapped. On the strength of Eq. (2), the condition $\Delta = 0$ is equivalent to $(\partial\theta/\partial\xi_1)(\partial\theta/\partial\xi_2) = 0$. The line $\partial\theta/\partial\xi_2 = 0$ is

$$\zeta^2 = -\frac{\cos\rho}{2}\frac{\operatorname{tg}\nu}{1 + \sin\rho\sin 2\nu},$$

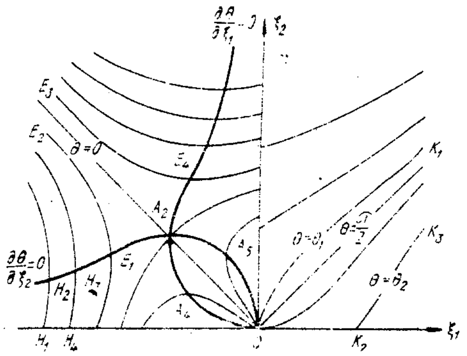

Fig. 1

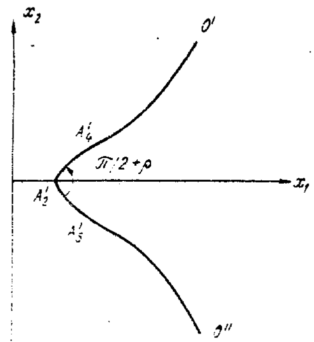

Fig. 2

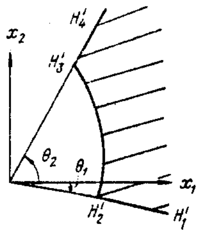

Fig. 3

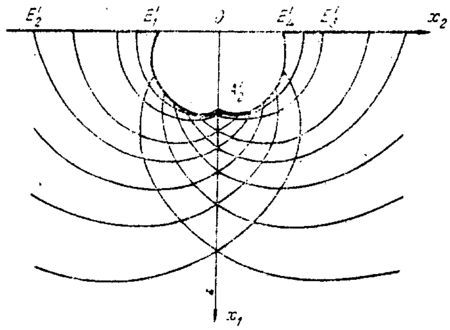

Fig. 4

and the line $\partial\theta/\partial\xi_2 = 0$ is

$$\zeta^2 = -\frac{\cos\rho}{2}\,\frac{\operatorname{ctg}\nu}{1+\sin\rho\sin 2\nu}.$$

The lines are symmetrical relative to the bisector of the second quadrant. Here (ζ, ν) are polar coordinates in the plane of parameters (ξ_1, ξ_2) (Fig. 1). In addition, for the mapping to be one-to-one, the increment to the angle θ in the region must be less than 2π. Figure 1 shows the lines θ = const in the (ξ_1, ξ_2) plane. We note that the solutions that are symmetrical in the plane of parameters relative to the line $\nu = \pi/4$ will be symmetrical relative to the Ox_2 axis of the physical plane. Analogously, the solutions that are symmetrical relative to the straight line $\nu = 3\pi/4$ in the physical plane will be symmetrical relative to the abscissa axis Ox_1.

It can also be shown that the line $\partial\theta/\partial\xi_i = 0$ $(i = 1, 2)$ in the (x_1, x_2) plane represents the set of cusps of the i-th family of slip lines and the envelope of the other family. For this reason, if the line $\Delta = 0$ is the boundary of the region being mapped, then the condition of limiting equilibrium, $\tau_\alpha = -\sigma_\alpha \tan\rho + k$, where σ_α, τ_α, are the normal and tangential stresses on the boundary, will be satisfied on it.

Let us examine the image of the region $OA_4A_2A_5O$ (see Fig. 1) in the physical plane (Fig. 2). Here the increments to θ are everywhere less than 2π. The image $O'A_4'A_2'A_5'O''$ has the form of a wedge angle $\pi/2 + \rho$. The section of the boundary $A_2'A_4'$ is transformed with convexity upwards and the section $A_4'O'$ is transformed with convexity downwards, and the solution is symmetrical relative to the Ox_1 axis. The condition of limiting equilibrium is satisfied on the boundary and at infinity $\varphi \to 0$, $\sigma \to -c_4$.

$$\sigma_{11} \to -c_4(1-\sin\rho) + k\operatorname{ctg}\rho,\ \sigma_{12} \to 0,\ \sigma_{22} \to -c_4\,(1+\sin\rho) + k\operatorname{ctg}\rho.$$

The region $H_1H_2H_3H_4$ (see Fig. 1), where H_1H_2 is the line $\theta = \theta_1 = \text{const}$ and H_3H_4 is the line $\theta = \theta_2 = \text{const}$, is mapped onto the wedge with angle $\theta_2 - \theta_1$ and cut-out vertex (Fig. 3).

The condition of limiting equilibrium is satisfied on the section of the boundary $H_2'H_3'$. This section, for abscissa points H_1, H_4 with sufficiently large modulus, is close to the arc of a logarithmic spiral, which approaches the arc of a circle for $\rho \to 0$.

We shall now examine the region $A_2E_1E_2E_3E_4A_2$ (see Fig. 1). Here E_1E_2 is the line $\theta = -\pi/2$ and E_3E_4 is the line $\theta = \pi/2$. The solution represents the half-plane with a cut-out (Fig. 4) and the section of the boundary $E_1'A_2'E_4'$ is the envelope of the slip lines. The sides of this section $E_1'A_2'$ and $A_2'E_4'$ bound regions shaped like little horns, bounded on the outside by the envelope and the inside by the slip line. Similar solutions (according to Hartman) for the case of perfect plasticity ($\rho = 0$) are described in [5].

At the point A_2', the boundary of the region has a break of magnitude $\pi/2 + \rho$. The maximum tangential stress is reached on the section $E_1'A_2'E_4'$. At infinity $\delta \to \rho/2 + \pi/4$, $\varphi \to \theta$, $\sigma \to -\infty$.

Finally, we shall examine the mapping of regions in the first quadrant, where $\Delta \neq 0$ everywhere. Only a single condition remains: the increment to θ in the region must be less than 2π. Let the region $OK_1K_3K_2O$ (see Fig. 1), whose boundaries are the lines $\theta = \text{const}$, be the region being mapped. Evidently, this region is mapped onto a wedge. At infinity $\sigma \to -c_4$, $\varphi \to \theta - \pi/2$.

LITERATURE CITED

1. V. V. Sokolovskii, Statics of Granular Media [in Russian], Fizmatgiz, Moscow (1960).
2. L. Prandtl, "Hardness of plastic materials and resistance to cutting," in: Theory of Plasticity [Russian translation], Inostr. Lit., Moscow (1948). Moscow (1948).
3. A. F. Revuchenko, "A class of exact solutions of the equations of ideal plasticity," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1975).
4. A. Nadai, "Über die Gleit und Verzweigungsflächen einiger Gleichgewichtszustäde Bildsamer Massen und die Nachspannungen Bleibend verzerrter Körper," Z. Phys., No. 30 (1924).
5. A. Nadai, Plasticity and Failure in Solids [Russian translation], Vol. 2, Mir, Moscow (1969).